

ABSOLUTE CONTINUITY OF BERNOULLI CONVOLUTIONS, A SIMPLE PROOF

YUVAL PERES AND BORIS SOLOMYAK

ABSTRACT. The distribution ν_λ of the random series $\sum \pm\lambda^n$ has been studied by many authors since the two seminal papers by Erdős in 1939 and 1940. Works of Alexander and Yorke, Przytycki and Urbański, and Ledrappier showed the importance of these distributions in several problems in dynamical systems and Hausdorff dimension estimation. Recently the second author proved a conjecture made by Garsia in 1962, that ν_λ is absolutely continuous for a.e. $\lambda \in (1/2, 1)$. Here we give a considerably simplified proof of this theorem, using differentiation of measures instead of Fourier transform methods. This technique is better suited to analyze more general random power series.

1. Introduction

Consider the random series

$$Y_\lambda = \sum_{n=0}^{\infty} \pm\lambda^n, \quad \text{for } 0 < \lambda < 1,$$

where the “+” and “−” signs are chosen independently with probability 1/2. Let ν_λ be the distribution of Y_λ :

$$\nu_\lambda(E) = \text{Prob}\{Y_\lambda \in E\}.$$

The measure ν_λ is the infinite convolution product of $\frac{1}{2}(\delta_{-\lambda^n} + \delta_{\lambda^n})$ for $n \geq 0$ and is sometimes called an “infinite Bernoulli convolution”. The Fourier transform of ν_λ can be represented as a convergent infinite product: $\hat{\nu}_\lambda(u) = \prod_{n=0}^{\infty} \cos(\lambda^n u)$.

Infinite Bernoulli convolutions have been studied since the 1930’s; we sketch some of the relevant background below. The following theorem was recently proved by the second author, who verified a conjecture of Garsia (1962). Our goal here is to present a self-contained simple proof of this result.

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Theorem 1 [Solomyak (1995)]. *For a.e. $\lambda \in (\frac{1}{2}, 1)$ the measure ν_λ is absolutely continuous and has an L^2 -density.*

The main simplification in the present proof is the replacement of the Fourier transform estimates in Solomyak (1995) by the differentiation method used by Mattila (1990) to prove projection theorems. This method is more powerful and yields some extensions of Theorem 1 (see Section 4). Heuristically, the measures ν_λ may be viewed as “nonlinear projections” of the uniform measure on sequence space; the nonlinearity is controlled by establishing a transversality property of certain power series. This property is closely related to the bounds on double zeros used in the original proof, but is easier to verify.

Our new proof of Theorem 1 is in sections 2–3. We now describe some of the previous work on infinite Bernoulli convolutions. Jessen and Wintner (1935) showed that ν_λ is either absolutely continuous or purely singular, depending on λ ; see Breiman (1968) for a proof using Kolmogorov’s 0-1 Law. Kershner and Wintner (1935) observed that ν_λ is singular for $\lambda \in (0, \frac{1}{2})$, since it is supported on a Cantor set of zero Lebesgue measure. Wintner (1935) noted that ν_λ is uniform on $[-2, 2]$ for $\lambda = 1/2$, and for $\lambda = 2^{-1/k}$ with $k \geq 1$ it is absolutely continuous, with a density having $k - 1$ derivatives.

For any $\lambda \in (0, 1)$, the closed support K_λ of ν_λ satisfies the equation

$$K_\lambda = (1 + \lambda K_\lambda) \cup (-1 + \lambda K_\lambda).$$

It is easy to show that there exists a *unique* compact set $K_\lambda \subset \mathbb{R}$ for which this equation holds (see, e.g. Hutchinson (1981)). If $\lambda \in (\frac{1}{2}, 1)$, then the interval $[-(1 - \lambda)^{-1}, (1 - \lambda)^{-1}]$ satisfies the same equation, and therefore it must coincide with K_λ . Thus one might surmise that ν_λ is absolutely continuous for all such λ . However, by examining the Fourier Transform, Erdős (1939) showed that ν_λ is singular when λ is the reciprocal of a PV (Pisot-Vijayaraghavan) number. Recall that a PV-number is an algebraic integer, all of whose conjugates are less than one in modulus; the golden ratio $(1 + \sqrt{5})/2$ is the best known example. No other $\lambda \in (\frac{1}{2}, 1)$ are known to give singular ν_λ .

In another direction, Erdős (1940) showed that ν_λ is absolutely continuous for a.e. $\lambda \in (a, 1)$ for some $a < 1$ (in his proof a is rather close to 1 and not given explicitly). Examples of algebraic $\lambda \in (\frac{1}{2}, 1)$ (other than $2^{-1/k}$) with absolutely continuous ν_λ were found by Garsia (1962). In the same paper, Garsia conjectured that absolute continuity holds for a.e. λ in the full interval $(\frac{1}{2}, 1)$. Kahane and Salem (1958) obtained a criterion for the measure ν_λ to have L^2 -density, but until recently it was not clear how

it could be applied. Applications of absolute continuity of ν_λ to dynamical systems and Hausdorff dimension were given by Alexander and Yorke (1984), Przytycki and Urbański (1989), and Ledrappier (1992).

2. Beginning of the proof

Let $B_r(x) = [x - r, x + r]$. Consider the lower derivative of ν_λ :

$$\underline{D}(\nu_\lambda, x) = \liminf_{r \downarrow 0} (2r)^{-1} \nu_\lambda[B_r(x)].$$

A standard application of the Vitali covering theorem shows that ν_λ is absolutely continuous if and only if $\underline{D}(\nu_\lambda, x) < \infty$ for ν_λ almost all $x \in \mathbb{R}$ (see, e.g. Mattila (1995) §2.12). If we show that

$$(1) \quad \mathcal{S} := \int_I \int_{\mathbb{R}} \underline{D}(\nu_\lambda, x) d\nu_\lambda(x) d\lambda < \infty,$$

then this criterion yields that ν_λ is absolutely continuous for a.e. $\lambda \in I$. The interval I will be specified later; see the last section for measurability. Notice that for λ such that ν_λ is absolutely continuous, $\underline{D}(\nu_\lambda, x)$ is a version of the Radon-Nikodym derivative $\frac{d\nu_\lambda(x)}{dx}$, so $\int_{\mathbb{R}} \underline{D}(\nu_\lambda, x) d\nu_\lambda(x) < \infty$ implies that $\frac{d\nu_\lambda(x)}{dx} \in L^2(\mathbb{R})$. By Fatou's Lemma,

$$(2) \quad \mathcal{S} \leq \liminf_{r \downarrow 0} (2r)^{-1} \int_I \int_{\mathbb{R}} \nu_\lambda[B_r(x)] d\nu_\lambda(x) d\lambda.$$

We need to restate the definition of ν_λ as the distribution of the random series Y_λ . Let $\Omega = \{-1, 1\}^{\mathbb{Z}_+}$ be the sequence space equipped with the product topology. Let μ be the Bernoulli measure on Ω with the weights $(1/2, 1/2)$. There is a natural map $\Pi_\lambda : \Omega \rightarrow \mathbb{R}$, given by

$$\Pi_\lambda(\omega) = \sum_{n=0}^{\infty} \omega_n \lambda^n.$$

A moment's reflection gives

$$(3) \quad \nu_\lambda = \mu \circ \Pi_\lambda^{-1}.$$

Now Π_λ is continuous so we can change variables in (2):

$$(4) \quad \mathcal{S} \leq \liminf_{r \downarrow 0} (2r)^{-1} \int_I \int_{\Omega} \nu_\lambda[B_r(\Pi_\lambda(\omega))] d\mu(\omega) d\lambda.$$

Next, denote by 1_A the characteristic function of a set A and use (3) again:

$$\nu_\lambda[B_r(\Pi_\lambda(\omega))] = \int_{\mathbb{R}} 1_{B_r(\Pi_\lambda(\omega))} d\nu_\lambda = \int_{\Omega} 1_{\{\tau \in \Omega: |\Pi_\lambda(\tau) - \Pi_\lambda(\omega)| \leq r\}} d\mu(\tau).$$

Substitute this into (4), exchange the order of integration, and integrate with respect to λ to obtain

$$(5) \quad \mathcal{S} \leq \liminf_{r \downarrow 0} (2r)^{-1} \int_{\Omega} \int_{\Omega} \mathcal{L}\{\lambda \in I : |\Pi_\lambda(\tau) - \Pi_\lambda(\omega)| \leq r\} d\mu(\tau) d\mu(\omega),$$

where \mathcal{L} is Lebesgue measure. Let

$$\phi_{\tau,\omega}(\lambda) = \Pi_\lambda(\tau) - \Pi_\lambda(\omega) = \sum_{n=0}^{\infty} (\tau_n - \omega_n) \lambda^n.$$

Notice that $\tau_n - \omega_n \in \{-2, 0, 2\}$. We need to estimate $\mathcal{L}\{\lambda \in I : |\phi_{\tau,\omega}(\lambda)| \leq r\}$. We can write

$$(6) \quad \phi_{\tau,\omega}(\lambda) = 2\lambda^k g(\lambda),$$

where $k = |\omega \wedge \tau| = \min\{n : \omega_n \neq \tau_n\}$ and g is a power series with constant term ± 1 . Without loss of generality, $\omega_k < \tau_k$ so that

$$(7) \quad g(x) = 1 + \sum_{n=1}^{\infty} b_n x^n, \quad \text{with } b_n \in \{-1, 0, 1\}.$$

In order to estimate the integrand in (5), assume that on the interval $I = [\lambda_0, \lambda_1]$ the following δ -transversality condition holds for some $\delta > 0$:

$$(8) \quad \text{For any } g \text{ of the form (7) and any } x \in I, \text{ if } g(x) < \delta \text{ then } g'(x) < -\delta.$$

The assumption means that the graph of each g as in (7) crosses transversally, with slope at most $-\delta$, all horizontal lines below height δ that it meets. In the next section we will find intervals where (8) holds. We claim that this assumption implies that for all g of the form (7) and all $\rho > 0$,

$$(9) \quad \mathcal{L}\{\lambda \in I : |g(\lambda)| \leq \rho\} \leq 2\delta^{-1}\rho.$$

This is obvious if $\rho \geq \delta$; otherwise, $g'(\lambda) < -\delta$ whenever $|g(\lambda)| \leq \rho$ by δ -transversality. Thus g is monotone decreasing on the set in (9) with $|g'| > \delta$, so the inequality (9) is proved.

By (6), $|\phi_{\tau,\omega}(\lambda)| \leq r$ implies that $|g(\lambda)| \leq \lambda_0^{-k}r/2$ for $\lambda \in I = [\lambda_0, \lambda_1]$. Applying (9) with $\rho = \lambda_0^{-k}r/2$, we obtain

$$\mathcal{L}\{\lambda \in I : |\phi_{\tau,\omega}(\lambda)| \leq r\} \leq \delta^{-1}\lambda_0^{-k}r.$$

Substituting this in (5) yields

$$\begin{aligned} \mathcal{S} &\leq \delta^{-1} \liminf_{r \downarrow 0} (2r)^{-1} \int_{\Omega} \int_{\Omega} \lambda_0^{-|\omega \wedge \tau|} r \, d\mu(\tau) \, d\mu(\omega) \\ &= (2\delta)^{-1} \sum_{k=0}^{\infty} \lambda_0^{-k} (\mu \times \mu) \{(\omega, \tau) : |\omega \wedge \tau| = k\} \tag{10} \\ &= (2\delta)^{-1} \sum_{k=0}^{\infty} \lambda_0^{-k} 2^{-k-1} < \infty, \end{aligned}$$

where the last step requires that $\lambda_0 > \frac{1}{2}$. This proves (1), so we have shown the absolute continuity of ν_λ for a.e. $\lambda \in I$ assuming δ -transversality.

3. Establishing transversality, and conclusion of the proof

Definition. A power series $h(x)$ is called a $(*)$ -function if for some $k \geq 1$ and $a_k \in [-1, 1]$,

$$h(x) = 1 - \sum_{i=1}^{k-1} x^i + a_k x^k + \sum_{i=k+1}^{\infty} x^i.$$

In Solomyak (1995) it was shown that among all convex combinations of series of the form (7), the power series with the smallest double zero must be a $(*)$ -function. Here we bypass this optimization argument and prove directly that an estimate on a $(*)$ -function and its derivative yield δ -transversality.

Lemma. *Suppose that a $(*)$ -function h satisfies*

$$(11) \quad h(x_0) > \delta \quad \text{and} \quad h'(x_0) < -\delta$$

for some $x_0 \in (0, 1)$ and $\delta \in (0, 1)$. Then the transversality condition (8) holds on $[0, x_0]$.

Proof. First observe that $h(x) > \delta$ and $h'(x) < -\delta$ for all $x \in [0, x_0]$. Indeed, h'' has at most one zero on $(0, 1)$ being a power series with at most one coefficient sign change. We have $h'(0) = -1$ if $k > 1$ and $h'(0) = a_1$ otherwise. Thus $h'(0) < -\delta$ (if $k = 1$ this follows from $h'(x_0) < -\delta$). Since

$h'(1-) = +\infty$, we must have $h'(x) < -\delta$ for all $x \in (0, x_0)$, otherwise h'' would have at least two zeros. Now obviously, $h(x) > h(x_0) > \delta$ for $x \in (0, x_0)$.

Let $g(x)$ be a power series of the form (7). Consider $f(x) = g(x) - h(x)$. Then (7) and the definition of a $(*)$ -function imply that $f(x) = \sum_{i=1}^l c_i x^i - \sum_{i=l+1}^\infty c_i x^i$, where $c_i \geq 0$ and $l = k-1$ or $l = k$. We have for any $x \in [0, x_0]$, by the claim proved above:

$$g(x) < \delta \Rightarrow f(x) < 0 \Rightarrow f'(x) < 0 \Rightarrow g'(x) < -\delta.$$

The middle implication is a consequence of one coefficient sign change:

$$\begin{aligned} f(x) < 0 &\Rightarrow \sum_{i=1}^l c_i x^i < \sum_{i=l+1}^\infty c_i x^i \Rightarrow \\ &\sum_{i=1}^l c_i i x^{i-1} < \sum_{i=l+1}^\infty c_i i x^{i-1} \Rightarrow f'(x) < 0. \quad \square \end{aligned}$$

Proof of Theorem 1 concluded: Observe that if ν_{λ^2} is absolutely continuous, then so is ν_λ , since it is a convolution of ν_{λ^2} with some other measure. Thus, it is enough to prove absolute continuity of ν_λ for a.e. $\lambda \in [\frac{1}{2}, 2^{-1/2}]$. First we establish that δ -transversality holds in the smaller interval $[\frac{1}{2}, 2^{-2/3}]$, and then indicate how to cover $[2^{-2/3}, 2^{-1/2}]$.

In order to apply the lemma, it remains to find an appropriate $(*)$ -function. Let

$$h(x) = 1 - x - x^2 - x^3 + 0.5x^4 + \sum_{i=5}^\infty x^i$$

(found by a simple computer search). This is a $(*)$ -function, satisfying $h(2^{-2/3}) > 0.07$ and $h'(2^{-2/3}) < -0.09$, so δ -transversality is verified for the interval $[0, 2^{-2/3}]$. The argument in Section 2 can be applied in $I = [\lambda_0, 2^{-2/3}]$ for any $\lambda_0 > \frac{1}{2}$, and this establishes absolute continuity for a.e. $\lambda \in [\frac{1}{2}, 2^{-2/3}]$.

Unfortunately, in this way we cannot cover $[\frac{1}{2}, 2^{-1/2}]$ since there is a power series of the form (7) with a double zero around 0.68 (see Solomyak (1995)), and having a double zero obviously contradicts δ -transversality. In order to cover the interval $[2^{-2/3}, 2^{-1/2}]$ consider the “thinned” random series $Z_\lambda = \sum_{i \neq 2+3j} \pm \lambda^i$, with every third term removed. If the distribution of Z_λ is absolutely continuous, then so is ν_λ . Replacing the series Y_λ

by Z_λ is equivalent to replacing the uniform measure μ on Ω by a measure $\tilde{\mu}$ on $\{-1, 0, 1\}^{\mathbb{Z}^+}$ according to which every third symbol is forced to be 0 and all other symbols are -1 or 1 with probability $1/2$.

The analogue of (10) with $\tilde{\mu}$ instead of μ goes through provided that $\lambda_0 > 2^{-2/3}$, since the $\tilde{\mu} \times \tilde{\mu}$ probability of having two sequences coincide up to the k -th term is $\sim 2^{-2k/3}$. Again, δ -transversality is needed but for a smaller class, namely, for power series of the form (7) that satisfy either $b_{3j+1} = 0$ for all $j \geq 0$, or $b_{3j+2} = 0$ for all $j \geq 0$. We claim that in both cases δ -transversality holds on $(0, 2^{-1/2})$. Indeed, the lemma has an obvious “thinned” version, and it suffices to check that the following two thinned $(*)$ -functions

$$h_1(x) = 1 - x - x^3 - x^4 + \sum_{j=2}^{\infty} (x^{3j} + x^{3j+1}),$$

$$h_2(x) = 1 - x^2 - x^3 - x^5 - x^6 + \sum_{j=3}^{\infty} (x^{3j-1} + x^{3j}),$$

satisfy the condition (11) at $x_0 = 2^{-1/2}$. \square

4. Concluding remarks

1. Strictly speaking, one needs to verify that the inner integral in (1) is measurable as a function of λ , before applying Fatou’s Lemma. Using (3) this inner integral can be written as

$$\int_{\Omega} \liminf_{r \downarrow 0} (2r)^{-1} \int_{\Omega} 1_{\{\tau \in \Omega: |\Pi_\lambda(\tau) - \Pi_\lambda(\omega)| \leq r\}} d\mu(\tau) d\mu(\omega),$$

and checking measurability in λ of the last expression is routine.

2. In the first part of the proof of Theorem 1, the criterion of Kahane and Salem (1958) could be used instead of Mattila’s approach used here.

3. We learned the idea of using transversality from Pollicot and Simon (1995).

4. It is natural to ask about the distribution of the random sum Y_λ when the signs are governed by a nonuniform probability measure $\tilde{\mu}$ on the sequence space Ω , instead of the uniform measure μ . The argument in section 2 is quite general, and shows that if the series (10) (with $\tilde{\mu}$ in place of μ) converges, then the corresponding distribution $\tilde{\nu}_\lambda$ has L^2 -density for a.e. λ in any interval $I = [\lambda_0, \lambda_1]$ where the δ -transversality condition holds. This result is sharp in the sense that if the series (10) with $\tilde{\mu}$ diverges, then for all $\lambda < \lambda_0$ the measure $\tilde{\nu}_\lambda$ cannot have an L^2 -density. In Peres and

Solomyak (1995), analogous conditions are given for $\tilde{\nu}_\lambda$ to have a density in L^q , with $q \in (1, 2)$. In general, the critical λ for different q are distinct, and depend on the multifractal spectrum of $\tilde{\mu}$. We illustrate these results in the case where $\tilde{\mu}$ is Bernoulli, which corresponds to the random series Y_λ with the signs chosen independently with probabilities $(p, 1 - p)$. Let ν_λ^p be its distribution.

Theorem 2. *Let I be an interval where the δ -transversality condition holds.*

(a) ν_λ^p is singular for all $\lambda < p^p(1 - p)^{1-p}$ and absolutely continuous for a.e. $\lambda > p^p(1 - p)^{1-p}$ in the interval I .

(b) Let $q \in (1, 2]$. The measure ν_λ^p does not have a density in L^q for all $\lambda < (p^q + (1 - p)^q)^{\frac{1}{q-1}}$, and has a density in L^q for a.e. $\lambda > (p^q + (1 - p)^q)^{\frac{1}{q-1}}$ in the interval I .

Mark Pollicott has informed us that Charmaine Leech, a graduate student at Warwick, has also been working in this direction.

5. As noted in Solomyak (1995), Theorem 1 immediately implies that for a.e. $\lambda \in [2^{-1/2}, 1]$, the density of ν_λ is continuous. It is unknown whether this also holds for a.e. $\lambda \in [\frac{1}{2}, 2^{-1/2}]$.

6. The absolute continuity of ν_λ in Theorem 1 is proven for a.e. $\lambda \in [\frac{1}{2}, 1]$. What can be said about the exceptional set? The known exceptions, the reciprocals of PV-numbers, form a countable set which is known to be closed and have the maximal (isolated) point ≈ 0.7548777 . Salem (1943) proved the following converse to a result of Erdős (1939): If $1/\lambda$ is not a PV-number then $\hat{\nu}_\lambda(u) \rightarrow 0$ as $u \rightarrow \infty$. As shown by Kahane (1971), the argument of Erdős (1940) implies that the Hausdorff dimension of the exceptional set in $(a, 1]$ tends to 0 as $a \rightarrow 1$. Perhaps a modification of our methods can give a bound on this Hausdorff dimension for all a , by analogy with a projection theorem of Falconer (1982).

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DEPARTMENT OF STATISTICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA .

Current address: Institute of Mathematics, The Hebrew University, Jerusalem, Israel

E-mail address: peres@math.huji.ac.il

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA
98195-4350

E-mail address: solomyak@math.washington.edu