Alternation makes the adversary weaker in two-player games

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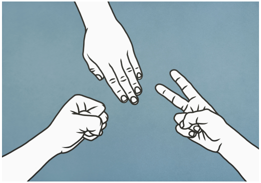
Normal Form Games

Rock-Paper-Scissors



Payoff matrix of Alice

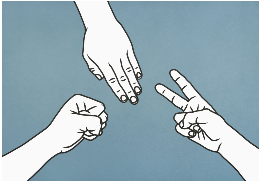
Alice/Bob	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Payoff matrix of Bob

Alice/Bob	Rock	Paper	Scissors
Rock	0	1	-1
Paper	-1	0	1
Scissors	1	-1	0

Normal Form Games

Rock-Paper-Scissors



Alice plays Rock

Alice/Bob	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Alice gets -1

Bob plays Scissors

Alice/Bob	Rock	Paper	Scissors
Rock	0	1	-1
Paper	-1	0	1
Scissors	1	-1	0

Bob gets 1

Normal Form Games

Normal-Form Games

Matrices A and B

Payoff matrix A of Alice

A/B	1	$\cdots$	j	$\cdots$	m
1	A_{11}	$\cdots$	A_{12}	$\cdots$	A_{1m}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
i	A_{i1}	$\cdots$	A_{ij}	$\cdots$	A_{im}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
n	A_{n1}	$\cdots$	A_{n2}	$\cdots$	A_{nm}

Payoff matrix B of Bob

A/B	1	$\cdots$	j	$\cdots$	m
1	B_{11}	$\cdots$	B_{12}	$\cdots$	B_{1m}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
i	B_{i1}	$\cdots$	B_{ij}	$\cdots$	B_{im}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
n	B_{n1}	$\cdots$	B_{n2}	$\cdots$	B_{nm}

Normal Form Games

Normal-Form Games

Matrices A and B

Alice plays action i

A/B	1	$\cdots$	j	$\cdots$	m
1	A_{11}	$\cdots$	A_{1j}	$\cdots$	A_{1m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	A_{i1}	$\cdots$	A_{ij}	$\cdots$	A_{im}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	A_{n1}	$\cdots$	A_{nj}	$\cdots$	A_{nm}

Alice gets A_{ij}

Bob plays action j

A/B	1	$\cdots$	j	$\cdots$	m
1	B_{11}	$\cdots$	B_{1j}	$\cdots$	B_{1m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	B_{i1}	$\cdots$	B_{ij}	$\cdots$	B_{im}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	B_{n1}	$\cdots$	B_{nj}	$\cdots$	B_{nm}

Bob gets B_{ij}

Normal Form Games

Normal-Form Games - Mixed Strategies

Matrices A and B

Alice plays a prob. distr.

$x = (x_1, \ldots, x_n) \in \Delta_n$

A/B	1	$\cdots$	j	$\cdots$	m
1	A_{11}	$\cdots$	A_{12}	$\cdots$	A_{1m}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
i	A_{i1}	$\cdots$	A_{ij}	$\cdots$	A_{im}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
n	A_{n1}	$\cdots$	A_{n2}	$\cdots$	A_{nm}

Bob plays a prob. distr.

$y = (y_1, \ldots, y_m) \in \Delta_m$

A/B	1	$\cdots$	j	$\cdots$	m
1	B_{11}	$\cdots$	B_{12}	$\cdots$	B_{1m}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
i	B_{i1}	$\cdots$	B_{ij}	$\cdots$	B_{im}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
n	B_{n1}	$\cdots$	B_{n2}	$\cdots$	B_{nm}

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$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
i	A_{i1}	$\cdots$	A_{ij}	$\cdots$	A_{im}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
n	A_{n1}	$\cdots$	A_{n2}	$\cdots$	A_{nm}

Alice's expected cost

$x^\top A y$

Bob plays a prob. distr.

$y = (y_1, \ldots, y_m) \in \Delta_m$

A/B	1	$\cdots$	j	$\cdots$	m
1	B_{11}	$\cdots$	B_{12}	$\cdots$	B_{1m}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
i	B_{i1}	$\cdots$	B_{ij}	$\cdots$	B_{im}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
n	B_{n1}	$\cdots$	B_{n2}	$\cdots$	B_{nm}

Bob's expected cost

$x^\top B y$

Normal Form Games

Normal-Form Games - Mixed Strategies

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Alice plays a prob. distr.

$x = (x_1, \ldots, x_n) \in \Delta_n$

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$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
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$$x^\top A y$$

Bob plays a prob. distr.

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$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
n	B_{n1}	$\cdots$	B_{n2}	$\cdots$	B_{nm}

$$x^\top B y$$

Normal-Form Games over Time

- Alice and Bob play the normal-form game (A, B) over T rounds.
  - *Simultaneous play*: Agents simultaneously select their strategies at each round (very well-studied).
  - *Alternating play*: Agents alternatingly update their strategies (this work).

Simultaneous Play

Simultaneous Play

Bob challenges Alice to play (A, B) for T rounds.

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Bob challenges Alice to play (A, B) for T rounds.

- Alice selects x_1 / Bob selects $y_1 \longrightarrow$ Alice gets $x_1^\top A y_1$

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Bob challenges Alice to play (A, B) for T rounds.

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- Alice selects x_2 / Bob selects y_2

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- Alice selects x_2 / Bob selects $y_2 \longrightarrow$ Alice gets $x_2^\top A y_2$
- Alice selects x_3 / Bob selects y_3

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- $\ldots$

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- $\ldots$

How shoud Alice select her actions over time?

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- $\ldots$

How shoud Alice select her actions over time? *No Regret algorithms*

- Regret Matching [Blackwell '65]
- Hedge [Freund et al. '97]
- Online Gradient Descent [Zinkevic '03]
- Follow the Regularized Leader [Abernethy et al. '10]
- ....

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Hedge Algorithm [Freund and Schapire '97] Godel Prize '03

$$x_{t+1}(i) = \frac{x_t(i) \cdot e^{-\gamma [A y_t]_i}}{\sum_{j=1}^n x_t(j) \cdot e^{-\gamma [A y_t]_j}}$$

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$$x_{t+1}(i) = \frac{x_t(i) \cdot e^{-\gamma [A y_t]_i}}{\sum_{j=1}^n x_t(j) \cdot e^{-\gamma [A y_t]_j}}$$

Theorem (Freund and Schapire JCSS '97)

No matter Bob's strategies $y_1, \ldots, y_T$, the regret of Alice $\mathcal{R}(T)$

$$\mathcal{R}(T) := \underbrace{\sum_{t=1}^T x_t^\top A y_t}_{\text{cost of Alice}} - \underbrace{\min_{i \in [n]} \sum_{t=1}^T [A y_t]_i}_{\text{cost of best action}} \leq \tilde{\mathcal{O}}\left(\sqrt{T}\right)$$

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- Alice selects x_3 / Bob selects $y_3 \longrightarrow$ Alice gets $x_3^\top A y_3$
- $\ldots$

Theorem (Freund and Schapire JCSS '97)

No matter Bob's strategies,

$$\underbrace{\frac{1}{T}\sum_{t=1}^{T} x_t^\top A y_t}_{\text{time-average cost}} \leq \underbrace{\frac{1}{T}\min_{x \in \Delta_n}\sum_{t=1}^{T} x^\top A y_t}_{\text{best fixed action}} + \tilde{\mathcal{O}}\left(\frac{\sqrt{T}}{T}\right) \to 0$$

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Can Alice do better?

Simultaneous play

Simultaneous Play

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Can Alice do better? No!

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Theorem (folkore)

Bob ***can always select*** *$y_1, \ldots, y_T$, the regret of Alice*

$$\mathcal{R}(T) \quad := \quad \underbrace{\sum_{t=1}^{T} x_t^\top A y_t}_{\text{cost of Alice}} \quad - \quad \underbrace{\min_{i \in [n]} \sum_{t=1}^{T} [A y_t]_i}_{\text{cost of best action}} \quad \geq \quad \Omega(\sqrt{T})$$

$\mathcal{O}(\sqrt{T})$ regret is the best Alice can get in **simultaneous play**!

Regret Lower Bound on Simultaneous play

- Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

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- No matter what $x_t \in \Delta_n$ Alice selects,

$$\mathbb{E}[x_t^\top A y_t] = 0 \text{ and thus } \sum_{t=1}^{T} \mathbb{E}[x_t^\top A y_t] = 0$$

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- However $\min_{i \in \{1,2\}} [A y_t]_i = -\Theta(\sqrt{T})$ since $[A y_t]_i = 1$ or -1 with probability $1/2$.

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- However $\min_{i \in \{1,2\}} [Ay_t]_i = -\Theta(\sqrt{T})$ since $[Ay_t]_i = 1$ or -1 with probability $1/2$.
- No matter how Alice her sequence $x_1, \ldots, x_T$:

$$\mathbb{E}\left[\sum_{t=1}^{T} x_t^\top A y_t - \min_{i \in [n]} \sum_{t=1}^{T} [Ay_t]_i\right] = \Theta(\sqrt{T})$$

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*Bob **can always select** $y_1, \ldots, y_T$, the regret of Alice*

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What if Alice and Bob play in alternating turns?

Solving Heads'up Poker

Online Learning + Simultaneous Play

Polaris [Bowling et al. AAMAS '09] → Decent Performance

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Online Learning + Alternating Play

Cepheus [Oskari et al. IJCAI '15], Libratus [Brown et al. IJCAI '17] → **Beat Human Experts!**

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Regret Guarantees → Faster Training!

Alternating Turns

Alternating Play

Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

Alternating Turns

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Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

- Bob selects y_0

Alternating Turns

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- Alice selects x_1

Alternating Turns

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Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

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- Alice selects $x_1 \longrightarrow$ Alice gets $x_1^\top A y_0$

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- Alice selects $x_1 \longrightarrow$ Alice gets $x_1^\top A y_0$
- Bob selects y_2

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- Bob selects $y_2 \longrightarrow$ Alice gets $x_1^\top A y_2$
- Alice selects x_3

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- Alice selects $x_3 \longrightarrow$ Alice gets $x_3^\top A y_2$
- ...

Can Alice have regret $\mathcal{R}(T)$ better than $\mathcal{O}(\sqrt{T})$?

$$\mathcal{R}(T) := \underbrace{\sum_{k=0}^{T/2} x_{2k+1}^\top A(y_{2k} + y_{2k+2})}_{\text{Alice's cost}} - \underbrace{\min_{x \in \Delta_n} \sum_{k=0}^{T/2} x^\top A(y_{2k} + y_{2k+2})}_{\text{best fixed action}}$$

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Previous $\Omega(\sqrt{T})$ lower bound fails!

Regret Lower Bound Fails!!!

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Regret Lower Bound Fails!!!

- Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- Bob is indifferent so it selects $y_t = (1, 0)$ or $(0, 1)$ with prob. $1/2$.
- Alice selects $x_t = (1, 0)$ if $y_{t-1} = (1, 0)$ and $x_t = (0, 1)$ if $y_{t-1} = (0, 1)$. Thus,

$$\sum_{t=1}^{T} \mathbb{E}[x_t^\top A(y_{t-1} + y_t)] = -T$$

Regret Lower Bound Fails!!!

- Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- Bob is indifferent so it selects $y_t = (1,0)$ or $(0,1)$ with prob. $1/2$.
- Alice selects $x_t = (1,0)$ if $y_{t-1} = (1,0)$ and $x_t = (0,1)$ if $y_{t-1} = (0,1)$. Thus,

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- However $\min_{i\in\{1,2\}}[A(y_t + y_{t-1})]_i = -\Theta(\sqrt{T})$ since $[Ay_t]_i = 1$ or -1 with probability $1/2$.
- No matter how Alice her sequence $x_1, \ldots, x_T$:

$$\mathbb{E}\left[\sum_{t=1}^{T} x_t^\top A(y_t + y_{t-1}) - \min_{i\in[n]} \sum_{t=1}^{T} [A(y_t + y_{t-1})]_i\right] = -T + \Theta(\sqrt{T})$$

Alternating Turns

Alternating Play

Bob challenges Alice to play (A, B) for T rounds in **alternating turns**.

- Bob selects y_0
- Alice selects $x_1 \longrightarrow$ Alice gets $x_1^\top A y_0$
- Bob selects $y_2 \longrightarrow$ Alice gets $x_1^\top A y_2$
- Alice selects $x_3 \longrightarrow$ Alice gets $x_3^\top A y_2$
- $\dots$

Can Alice have regret $\mathcal{R}(T)$ better than $\mathcal{O}(\sqrt{T})$?

$$\mathcal{R}(T) := \underbrace{\sum_{k=0}^{T/2} x_{2k+1}^\top A(y_{2k} + y_{2k+2})}_{\text{Alice's cost}} - \underbrace{\min_{x \in \Delta_n} \sum_{k=0}^{T/2} x^\top A(y_{2k} + y_{2k+2})}_{\text{best fixed action}}$$

Previous $\Omega(\sqrt{T})$ lower bound fails!

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Theorem (Skoulakis et al., NeurIPS '23 spotlight])

In alternating play*, Alice can always guarantee*

- $\tilde{\mathcal{O}}(T^{1/3})$ *regret for general games.*
- $\mathcal{O}(\log T)$ *for* $n = 2$ *actions (different algorithm).*

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Previous Results

- If **both** Alice and Bob use *Gradient Descent* in **unconstrained zero-sum games** $\longrightarrow \mathcal{O}(1)$ regret [Bailey et al. COLT 2020]
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Our algorithm (general simplex)

$$x_{2k+1} \leftarrow \mathrm{argmin}_{x \in \Delta_n} \left[3 \underbrace{x^\top A y_{2k}}_{\text{exploits } y_{2k}} + \underbrace{2 \sum_{k'=0}^{k-1} x^\top A y_{2k'}}_{\text{reinforces good past actions}} - \underbrace{\gamma \cdot \sum_{i=1}^{n} \log x_i}_{\text{prevents overfitting}} \right]$$

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Setting $\gamma = \mathcal{O}(T^{1/3}) \rightarrow \mathcal{O}(T^{1/3})$ alternating regret

Our Algorithm (General Case)

Algorithm 1: Our method

for rounds $t = 1, \ldots, T$ **do**

Alice *selects* sets $c_t \to Ay_t$ and selects the mixed strategy $x_t \in \Delta_2$,

$$x_t := \mathrm{argmin}_{x \in \Delta_n} \left[2\gamma c_{t-1}^\top x + \gamma \sum_{s=1}^{t-1} (c_s + c_{s-1})^\top x \underbrace{- \sum_{i=1}^{2} \log x_i / \gamma}_{\text{log-barrier penalty}} \right]$$

Alice **suffers** cost $(c_t + c_{t-1})^\top x_t$

end for

Analysis

Be the Regularized Leader

Let $y_{t+1} := \min_{x \in \Delta_n} \left[\gamma(c_t + c_{t-1})^\top x + \gamma \sum_{s=1}^{t-1} (c_s + c_{s-1})^\top x - \sum_{i=1}^{2} \log x_i\right]$.
Then,

$$\sum_{t=1}^{T} (c_t + c_{t-1})^\top \cdot y_{t+1} - \min_{x \in \Delta_n} \sum_{t=1}^{T} (c_t + c_{t-1})^\top \cdot x \leq \mathcal{O}\left(\frac{\log T}{\gamma}\right)$$

- **Regret Decomposition**

$$\sum_{t=1}^{T} (c_{t-1} + c_t)^\top (x_t - x^\star)$$
$$= \underbrace{\sum_{t=1}^{T} (c_{t-1} + c_t)^\top (y_{t+1} - x^\star)}_{\leq \mathcal{O}(\log T/\gamma)} + \sum_{t=1}^{T} (c_{t-1} + c_t)^\top (x_t - y_{t+1})$$

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Analysis

In order to control $\sum_{t=1}^{T}(c_{t-1}+c_t)^\top(x_t-y_{t+1})$, we notice that

- $\underbrace{y_{t+1} := \min_{x\in\Delta_2}\left[\gamma(c_t+c_{t-1})^\top x + \gamma\sum_{s=1}^{t-1}(c_s+c_{s-1})^\top x - \sum_{i=1}^{2}\log x_i\right]}_{\text{Be the Leader}}$
- $\underbrace{x_t := \min_{x\in\Delta_2}\left[2\gamma c_{t-1}^\top x + \gamma\sum_{s=1}^{t-1}(c_s+c_{s-1})^\top x - \sum_{i=1}^{2}\log x_i\right]}_{\text{our Algorithm}}$

Lemma (Skoulakis et al., 2023)

Let $x_t, y_{t+1} \in \Delta_n$ defined as above. Then,

$$x_t - y_{t+1} = \gamma \cdot A(x_t, y_{t+1})(c_t - c_{t-1})$$

where $A(x,y)$ is an $n\times n$ matrix with

$$||A(x,y) - A(x',y')||_2 \leq \mathcal{O}(1)\cdot||(x,y)-(x',y')||_2.$$

Analysis

- **Regret Decomposition**

$$
\begin{aligned}
&\sum_{t=1}^{T}(c_{t-1}+c_t)^\top(x_t-x^\star)\\
&\quad=\underbrace{\sum_{t=1}^{T}(c_{t-1}+c_t)^\top(y_{t+1}-x^\star)}_{\leq\mathcal{O}(\log T/\gamma)}+\underbrace{\sum_{t=1}^{T}(c_{t-1}+c_t)^\top(x_t-y_{t+1})}_{\text{upper bound this!}}\\
&\quad=\mathcal{O}(\log T/\gamma)+\gamma\sum_{t=1}^{T}(c_{t-1}+c_t)^\top A(x_t,y_{t+1})(c_t-c_{t-1})\\
&\quad=\mathcal{O}(\log T/\gamma)+\gamma\sum_{t=1}^{T}c_t^\top\left(A(x_t,y_{t+1})-A(x_{t-1},y_t)\right)c_t\\
&\quad=\mathcal{O}(\log T/\gamma)+\gamma\sum_{t=1}^{T}||c_t||_2^2\cdot||(x_t,y_{t+1})-(x_{t-1},y_t)||_2\\
&\quad=\mathcal{O}(\log T/\gamma)+\mathcal{O}(\gamma^2T)=\mathcal{O}(T^{1/3})\text{ for }\gamma=T^{-1/3}.
\end{aligned}
$$

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- If Alice admits 2 actions $\rightarrow \mathcal{O}(\log T)$ in alternating play!

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  - If Greedy BR (w_k) admits $\mathcal{O}(\sqrt{T})$ regret $\rightarrow$ FTRL (z_k) admits $\mathcal{O}(1)$ regret
- Alice plays x_{2k+1} in the convex hull of z_k and w_k.

Take-Away Message

Algorithms beyond $\mathcal{O}(\sqrt{T})$ regret lower bounds of simultaneous play.

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Thank you!!